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Real space renormalisation group for directed systems in arbitrary dimensions

J Kamphorst Leal da Silva† and M Droz

Département de Physique Théorique, Université de Genève, CH-1211 Genève 4, Switzerland

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Abstract. A real space renormalisation group approach to intrinsically anisotropic systems (i.e. systems having two different correlation lengths) is defined. The problems of directed self-avoiding walks (DSAW) and directed percolation in d dimensions are discussed in detail. For DSAW, one obtains the exact critical fugacity in all dimensions. For directed percolation, the percolation threshold p_c obtained is very good in two dimensions, asymptotically exact for large dimensions and quite good in between. The parallel and perpendicular correlation length exponents $\nu_\parallel$ and $\nu_\perp$ are computed for the two problems.

1. Introduction

Real space renormalisation group (RSRG) techniques have been extensively used in the past years to study critical phenomena (Burkhardt and van Leeuwen 1982). These approaches have been reasonably successful for both thermal and geometrical phase transitions providing that one length scale, the correlation length, dominates near criticality.

However, there is a class of problems for which no good RSRG exist, namely the problems characterised by two length scales. One encounters such a situation near a Lifshitz point (Hornreich *et al* 1975), in critical dynamics and in the so-called directed geometrical phase transitions. The problems of the directed self-avoiding walk (DSAW) and directed percolation belong to this class. These problems are intrinsically anisotropic. Two different correlation lengths, the parallel and transverse ones, have to be introduced associated with the two critical exponents $\nu_\parallel$ and $\nu_\perp$. The ratio $\theta = \nu_\parallel / \nu_\perp$ is called the anisotropy exponent.

Usual RSRG approaches use an isotropic rescaling factor b. Even if a reasonable phase diagram can be obtained, one does not know how to extract unambiguously the correlation length exponents (Redner and Yang 1982, de Oliveira 1983). To do that, one must use two different rescaling factors $b_\parallel$ and $b_\perp$ in the RSRG transformation in such a way that $b_\parallel = b_\perp^\theta$ (Phani and Dhar 1982). This implies that the shape of the cell must change at each renormalisation step. Note moreover that for small sizes only a restricted number of particular values for θ is realisable, limiting the feasibility of the method. The problem of asymmetric rescaling can possibly be avoided if one of the two parameters $b_\parallel$ and $b_\perp$ becomes infinite. This is realised in the so-called phenomenological renormalisation approach (Nadal *et al* 1982, Kinzel and

† On leave from the Departamento de Fisica, Universidade Federal de Minas Gerais, CP 702, 3000 Belo Horizonte, Minas Gerais, Brazil.

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Yeomans 1981) in which transfer matrix techniques are combined with finite-size scaling. This approach is mostly restricted in practice to the two-dimensional case. Another way to avoid the problem of changing cell shapes was proposed by Herrmann *et al* (1983). By patching square cells together to form an infinite sequence, they were able to obtain the values for the exponents ν of the two-dimensional directed percolation in the limit of an infinite rescaling factor.

A different approach, using an isotropic RSRG transformation, was introduced by Zhang and Yang (1984). The anisotropy is introduced into the problem by defining two effective lengths $L_{\parallel}$ and $L_{\perp}$ for both the original and renormalised lattice. These effective lengths are taken as the basic units for the measurement of the longitudinal and transverse correlation lengths. The anisotropy exponent θ is related to the anisotropic rescaling of $L_{\parallel}$ and $L_{\perp}$. It has been shown (Zhang and Yang 1984) that this method, implemented by an exact RSRG transformation, gave the exact results for the two-dimensional DSAW on a square lattice in the limit of an infinite rescaling factor.

However, for two-dimensional directed percolation, Zhang and Yang (1984) were not able to find sensible results. The difficulty is that one cannot devise an exact RSRG for this problem. If the approximate transformation used is too crude, it is impossible to extract meaningful exponents.

Recently, a new isotropic RSRG transformation was proposed by Kamphorst Leal da Silva and Droz (1985) for directed percolation. The merits of this transformation is to give almost the exact value for the percolation threshold in two dimensions. Moreover, this RSRG transformation can be defined in arbitrary dimension. For all dimensions, the prediction for the percolation threshold is within a few per cent of the best estimate. Finally, this transformation is obtained analytically for arbitrary rescaling factor b. The best results are obtained by considering the analytical continuation $b \to 1$. The main problem with this renormalisation transformation is how to compute the anisotropy exponent θ. It is thus legitimate to implement this renormalisation transformation by the two effective lengths (TEL) method of Zhang and Yang.

The paper is organised as follows. In § 2, we apply our RSRG transformation to the simpler problem of the directed self-avoiding walk and extract the correlation length exponents using the TEL method. Only the two-dimensional cases are discussed. The general d-dimensional case is considered in appendix 1. Section 3 is devoted to directed percolation. The two-dimensional case is treated in detail. Both the so-called isotropic (i.e. $p_x = p_y = p$) directed percolation and the strongly biased case ($p_x = p$, $p_y = 1$) are considered. The generalisation to d dimensions is treated in appendix 2. Finally the results are critically discussed in § 4. A test of self-consistency is discussed, allowing us to find a better anisotropy exponent.

2. Directed self-avoiding walk

The problem of the directed self-avoiding walk is the simplest directed geometrical problem one can think of. Let us consider as an illustration the two-dimensional case on a square lattice (see figure 1). Starting from the origin, one can move to the right or downward. Let p_x and p_y be the fugacities associated to a step to the right and downward respectively. The simplest case is the symmetric one for which $p_x = p_y$. For p_x larger than a threshold value, p_x^c, the walk will, on the average, extend to an arbitrary large distance. The problem is anisotropic in the sense that two particular directions

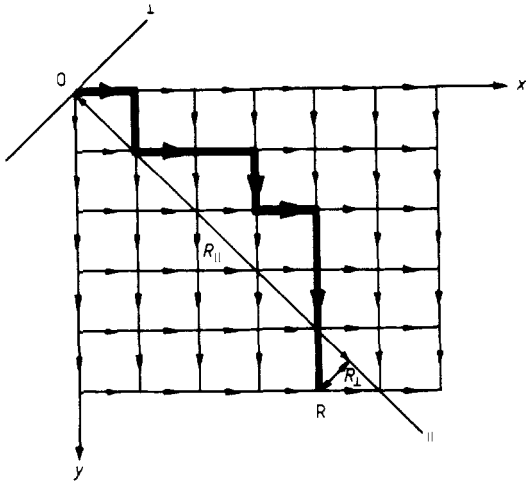

Figure 1. Directed self-avoiding walk on a two-dimensional square lattice. $R_{\parallel}$ and $R_{\perp}$ are the parallel and perpendicular projections of the walk OR.

can be defined, the parallel and perpendicular ones. When p_x approaches p_x^c, the mean longitudinal displacement $\langle R_{\parallel}\rangle$ and the root mean square perpendicular displacement $\langle R_{\perp}^2\rangle^{1/2}$ diverge as a power law, namely

$$\langle R_{\parallel}\rangle \sim (p_x - p_x^c)^{-\nu_{\parallel}}$$
$$\langle R_{\perp}^2\rangle^{1/2} \sim (p_x - p_x^c)^{-\nu_{\perp}}. \tag{2.1}$$

This problem and its d-dimensional generalisation are easily solvable by transfer matrix techniques (Redner and Majid 1983). In two dimensions, one finds $p_x^c = \frac{1}{2}$; $\nu_{\parallel} = 1$, corresponding to a ballistic-like propagation in the parallel direction and $\nu_{\perp} = \frac{1}{2}$, corresponding to a random walk propagation in the perpendicular direction. It is straightforward to write an exact RSRG transformation for this problem (Zhang and Yang 1984). Having in mind the application of the method to directed percolation and following our previous work (Kamphorst Leal da Silva and Droz 1985), we introduce a Migdal-like RSRG transformation for the DSAW. Let us first consider the two-dimensional case. The renormalisation transformation is defined as follows.

Consider a cell of size $b \times b$. p_x and p_y are the fugacities associated to a step in the x and y direction respectively (see figure 2(a)). The first step of the transformation consists in suppressing horizontal links in the cell by introducing a renormalised fugacity $\tilde{p}_x$ which guarantees that the fugacity that a walk starting at the origin and

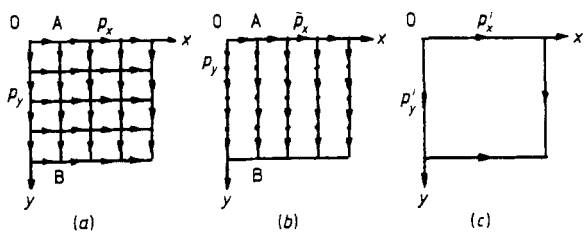

Figure 2. Successive steps of the renormalisation group transformation (here $b = 4$).

reaching a point on the line AB is conserved. Thus

$$\tilde{p}_x = p_x + p_x p_y + p_x p_y^2 + \cdots + p_x p_y^{b-1} = p_x(1-p_y^b)/(1-p_y). \tag{2.2}$$

The second step is to replace the fugacity p_y associated to a step in the y direction, by the fugacity $\tilde{p}_y$ associated to b steps in the y direction. This 'decimation' procedure leads to

$$\tilde{p}_y = p_y^b. \tag{2.3}$$

The third step is to repeat the first step for y, leading by analogy to

$$p'_y = \tilde{p}_y(1-\tilde{p}_x^b)/(1-\tilde{p}_x). \tag{2.4}$$

The last step is to 'decimate' along the x direction. Thus

$$p'_x = \tilde{p}_x^b. \tag{2.5}$$

This transformation is defined for all values of b. The best results are obtained by considering the analytic continuation $b = 1 + \delta b \to 1$. In this limit, one finds

$$\begin{aligned} p'_x &= p_x + \delta b\left(p_x \ln p_x - \frac{p_x p_y \ln p_y}{1-p_y}\right) \\ p'_y &= p_y + \delta b\left(p_y \ln p_y - \frac{p_x p_y \ln p_x}{1-p_x}\right). \end{aligned} \tag{2.6}$$

For the symmetric case $p_x = p_y$, the critical fixed point is $p^* = \frac{1}{2}$, i.e. the exact value.

One can then linearise the recursion relation (2.6) near the critical fixed point and we find

$$\left.\frac{\mathrm{d}p'_x}{\mathrm{d}p_x}\right|_{p^*=\frac{1}{2}} = 1 + \delta b 2 \ln 2 \equiv b^{1/\nu}. \tag{2.7}$$

Thus

$$\nu = 1/2 \ln 2 = 0.721. \tag{2.8}$$

Equation (2.7) defines formally a critical exponent ν, but is is not clear to which diverging length this exponent is associated.

Let us now introduce the two effective lengths method (TEL). For the original lattice, one decomposes a directed one-step walk into two components, along the parallel and the perpendicular directions. The effective lengths are simply the lengths of these two components. Assuming that the lattice constant of the original lattice is unity, and making the first step in the x direction, one has $L_\parallel(x, b=1) = 2^{-1/2} = L_\perp(x, b=1)$.

For the renormalised lattice, the situation is more complicated. Let us consider the first step of the renormalisation transformation. The two effective lengths $L_\parallel(x, b)$ and $L_\perp(x, b)$ are defined as follows. Consider all the paths Γ_i contributing to the first step of the renormalisation group transformation. Let $\bar{\mathbf{R}}_i$ be the endpoint of the path Γ_i and $\overline{\mathrm{OR}}_i$ the vector going from the origin to the point $\bar{\mathbf{R}}_i$. We can project $\overline{\mathrm{OR}}_i$ onto the parallel and perpendicular directions. Then $L_\parallel^1(x, b)$ and $L_\perp^1(x, b)$ are defined as the average, computed at the fixed point, over all the paths contributing to the two projections defined above. Moreover, in the last step of the renormalisation

transformation, one performs a decimation on $\tilde{p}_x$, leading to an extra multiplicative factor of b to the effective length computed in the first step. Thus we have

$$L_{\parallel}^{1}(p_x; b) = \frac{1}{\tilde{p}_x}\left(p_x \frac{\sqrt{2}}{2} + p_x p_y \sqrt{2} + \cdots\right) = \frac{\sqrt{2}}{2} \sum_{n=0}^{b-1} (n+1) p_y^n \frac{p_x}{\tilde{p}_x}. \tag{2.9}$$

But

$$\sum_{n=0}^{b-1} n p_y^n \overset{b=1+\delta b}{=} -\delta b \left(\frac{p_y}{1-p_y} + \frac{p_y \ln p_y}{1-p_y} + \frac{p_y^2 \ln p_y}{(1-p_y)^2}\right) \tag{2.10}$$

and

$$\sum_{n=0}^{b-1} p_y^n \overset{b=1+\delta b}{=} 1 - \delta b \frac{p_y \ln p_y}{1-p_y}. \tag{2.11}$$

Thus (2.9) becomes, using the result (2.2) for $\tilde{p}_x$:

$$L_{\parallel}^{1}(p_x; 1+\delta b) = \frac{\sqrt{2}}{2}\left[1 - \delta b \left(\frac{p_y}{1-p_y} + \frac{p_y \ln p_y}{(1-p_y)^2}\right)\right] \tag{2.12}$$

and finally after the 'decimation'

$$L_{\parallel}(p_x; 1+\delta b) = \frac{\sqrt{2}}{2}\left[1 - \delta b \left(\frac{p_y}{1-p_y} + \frac{p_y \ln p_y}{(1-p_y)^2} - 1\right)\right]. \tag{2.13}$$

For the perpendicular direction, one finds, along the same lines,

$$\begin{aligned} L_{\perp}^{1}(p_x; b) &= \frac{1}{\tilde{p}_x}\left(-\frac{\sqrt{2}}{2} p_x + \frac{\sqrt{2}}{2} p_x p_y^2 + \sqrt{2} p_x p_y^3 + \cdots\right) \\ &= \frac{1}{\tilde{p}_x} \frac{\sqrt{2}}{2} p_x \sum_{n=0}^{b-1} (n-1) p_y^n. \end{aligned} \tag{2.14}$$

Using (2.2), (2.10) and (2.11), one obtains

$$L_{\perp}^{1}(p_x; 1+\delta b) = -\frac{\sqrt{2}}{2}\left[1 + \delta b \left(\frac{p_y}{1-p_y} + \frac{p_y \ln p_y}{(1-p_y)^2}\right)\right] \tag{2.15}$$

and finally, after the 'decimation',

$$L_{\perp}(p_x; 1+\delta b) = -\frac{\sqrt{2}}{2}\left[1 + \delta b \left(1 + \frac{p_y}{1-p_y} + \frac{p_y \ln p_y}{(1-p_y)^2}\right)\right]. \tag{2.16}$$

Note that, instead of making the first step in the x direction, one can make the first step in the y direction and compute along the same line $L_{\parallel}(y, b)$ and $L_{\perp}(y, b)$. One can check that the results are similar to the one of equations (2.13) and (2.16) providing that one interchanges p_x and p_y in the formulae.

Let us now return to the TEL strategy. The two effective lengths $L_{\parallel}(x, b)$ and $L_{\perp}(x, b)$ are, in the critical domain, the distances made by one step of the directed walk relative to the preferred direction of the system. This means that $L_{\parallel}(x, b)$ and $L_{\perp}(x, b)$ form the basic units of the renormalised lattice. More explicitly, we can relate the two effective lengths to the anisotropic rescaling factors $b_{\parallel}$ and $b_{\perp}$ as follows:

$$\begin{aligned} L_{\parallel}(p_x'; b) &= b_{\parallel}\sqrt{2}/2 \\ L_{\perp}(p_x'; b) &= b_{\perp}\sqrt{2}/2. \end{aligned} \tag{2.17}$$

Thus the renormalised lattice is a deformed one with a parallelepipedic shape and having the same origin as the original lattice (see figure 3). Since the system remains unchanged, except for a rescaling, before and after the transformation, one can write

$$\frac{\xi_{\parallel}(p_x)}{\xi_{\parallel}(p'_x)} = \frac{L_{\parallel}(p_x^*; b)}{L_{\parallel}(p_x^*; 1)} \tag{2.18}$$

and

$$\frac{\xi_{\perp}(p_x)}{\xi_{\perp}(p'_x)} = \frac{L_{\perp}(p_x^*; b)}{L_{\perp}(p_x^*; 1)}. \tag{2.19}$$

Moreover

$$\begin{aligned} \xi_{\parallel}(p_x) &\sim |p_x - p_x^c|^{-\nu_{\parallel}} \\ \xi_{\perp}(p_x) &\sim |p_x - p_x^c|^{-\nu_{\perp}}. \end{aligned} \tag{2.20}$$

Thus

$$\left(\frac{\mathrm{d}p'_x}{\mathrm{d}p_x}\bigg|_{p_x^*}\right)^{\nu_{\parallel}} = \frac{L_{\parallel}(p_x^*; b)}{L_{\parallel}(p_x^*; 1)} \tag{2.21}$$

and

$$\left(\frac{\mathrm{d}p'_x}{\mathrm{d}p_x}\bigg|_{p_x^*}\right)^{\nu_{\perp}} = \frac{L_{\perp}(p_x^*; b)}{L_{\perp}(p_x^*; 1)}. \tag{2.22}$$

From (2.6), one finds, for $p_x = p_y = p^* = \frac{1}{2}$,

$$\frac{\mathrm{d}p'_x}{\mathrm{d}p_x}\bigg|_{p_x^*} = 1 + \delta b 2 \ln 2. \tag{2.23}$$

Thus using (2.13) and (2.16), one obtains

$$\nu_{\parallel} = 1 \qquad \nu_{\perp} = \frac{1 - \ln 2}{\ln 2} = 0.443. \tag{2.24}$$

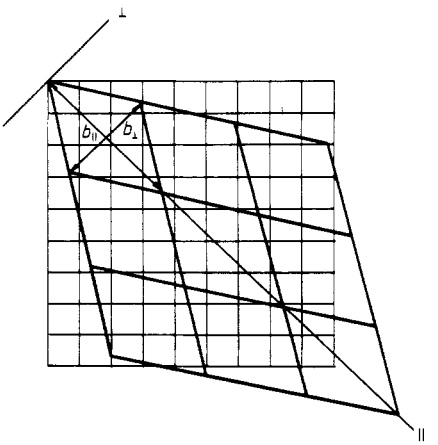

Figure 3. The renormalised lattice (bold lines) is deformed relative to the original lattice. $b_{\parallel}$ and $b_{\perp}$ are the parallel and transverse rescaling factors, respectively.

These results have to be compared with the exact values $\nu_{\parallel}=1$ and $\nu_{\perp}=0.5$. If the prediction for $\nu_{\parallel}$ is exact, the value obtained for $\nu_{\perp}$ is within about 10%.

This discrepancy could be due somewhat to the arbitrariness in the definition of $L_{\parallel}$ and $L_{\perp}$. Indeed, those effective lengths were defined by averaging the effective lengths of each path. It may happen that the effective length associated to some path is negative, leading to cancellations in the average (see equation (2.14) for example). A different way to proceed is to define the effective lengths as the square root of the average of the square effective lengths associated to the paths (Zhang and Yang 1984). Let us study this procedure for the DSAW. Following the previous notation, one has

$$(L_{\parallel}^{1}(p_x;b))^2=\frac{1}{\tilde{p}_x}\left(\frac{p_x}{2}+2p_xp_y+\frac{9}{2}p_xp_y^2+\ldots+\frac{b^2}{2}p_xp_y^{b-1}\right)$$
$$=\frac{1}{\tilde{p}_x}\left(\frac{p_x}{2}\sum_{n=0}^{b-1}n^2p_y^n+p_x\sum_{n=0}^{b-1}np_y^n+\tfrac{1}{2}p_x\sum_{n=0}^{b-1}p_y^n\right). \tag{2.25}$$

Using (2.2), (2.10), (2.11) and the result

$$\sum_{n=0}^{b-1}n^2p_y^n \overset{b=1+\delta b}{=} -\delta b\left(\frac{2p_y}{(1-p_y)^2}+\frac{p_y\ln p_y}{(1-p_y)^2}+\frac{2p_y^2\ln p_y}{(1-p_y)^3}\right) \tag{2.26}$$

and taking into account the multiplicative factor b^2 due to the 'decimation', one finds

$$L_{\parallel}^{2}(p_x;1+\delta b)=\frac{1}{2}\left[1-\delta b\left(\frac{2p_y}{1-p_y}+\frac{2p_y}{(1-p_y)^2}+\frac{3p_y\ln p_y}{(1-p_y)^2}+\frac{2p_y^2\ln p_y}{(1-p_y)^3}-2\right)\right]. \tag{2.27}$$

At the fixed point one finds

$$\frac{L_{\parallel}^{2}(p_x^*;1+\delta b)}{L_{\parallel}^{2}(p_x^*;1)}=1+\delta b(10\ln 2-4). \tag{2.28}$$

But now

$$\frac{L_{\parallel,\perp}^{2}(p_x;1+\delta b)}{L_{\parallel,\perp}^{2}(p_x;1)}=\frac{\mathrm{d}p_x'}{\mathrm{d}p_x}\bigg|_{p_x^*}^{2\nu_{\parallel,\perp}} \tag{2.29}$$

leading with (2.23) to $\nu_{\parallel}=1.057$. Along the same line, one can compute $L_{\perp}(p_x,b)$, namely

$$(L_{\perp}^{1}(p_x;b))^2=\frac{1}{\tilde{p}_x}(\tfrac{1}{2}p_x+\tfrac{1}{2}p_xp_y^2+2p_xp_y^3+\ldots+\tfrac{1}{2}(b-2)^2p_xp_y^{b-1}). \tag{2.30}$$

In the limit $b\to 1+\delta b$, using (2.2), (2.10), (2.11) and (2.26), and taking into account the multiplicative factor b^2 arising from the decimation, one finds

$$L_{\perp}^{2}(p_x;1+\delta b)=\frac{1}{2}\left[1+\delta b\left(2+\frac{2p_y}{(1-p_y)}-\frac{2p_y}{(1-p_y)^2}+\frac{2p_y\ln p_y}{(1-p_y)}\right.\right.$$
$$\left.\left.-\frac{p_y\ln p_y}{(1-p_y)^2}+\frac{2p_y^2\ln p_y}{(1-p_y)^2}-\frac{2p_y^2\ln p_y}{(1-p_y)^3}\right)\right]. \tag{2.31}$$

At the fixed point

$$\frac{L_{\perp}^{2}(p_x^*;1+\delta b)}{L_{\perp}^{2}(p_x^*;1)}=1+2\delta b\ln 2 \tag{2.32}$$

leading with (2.23) and (2.29) to $\nu_{\perp}=\frac{1}{2}$.

Thus, this second prescription, which is intuitively more satisfactory than the first one for the perpendicular direction, leads indeed to the exact transverse exponent $\nu_\perp$. On the other hand, the parallel exponent $\nu_\parallel$ is no longer exact but about 6% from its exact value.

Nevertheless, we see that the Migdal-like RSRG transformation used, implemented by the TEL procedure, allows for a very reasonable description of the DSAW problem.

The above derivation was restricted to the two-dimensional case. However, this method can be extended to arbitrary dimensions. This is a natural extension of the two-dimensional case but still a tedious exercise. The main steps of it and the results are given in appendix 1.

3. Directed bond percolation

The problem of isotropic directed bond percolation can be defined as follows. Let us consider a d-dimensional hypercubic lattice Λ^d. To each edge of the lattice a bond is either present with probability p or absent with probability $(1-p)$. Each bond is oriented, i.e. carries an arrow pointing in the direction of the cartesian axis parallel to the edge. One aim of the theory is to describe the properties of the connected pieces of the network. On the average, one can link the origin of the lattice 0 to the points within the domain sketched in figure 4. This domain is characterised by two correlation lengths: the one along the main diagonal of the lattice $\xi_\parallel$ and the one perpendicular to the main diagonal $\xi_\perp$. p_c is the directed bond percolation threshold. $\nu_\parallel$ and $\nu_\perp$ are the longitudinal and transverse correlation length exponents.

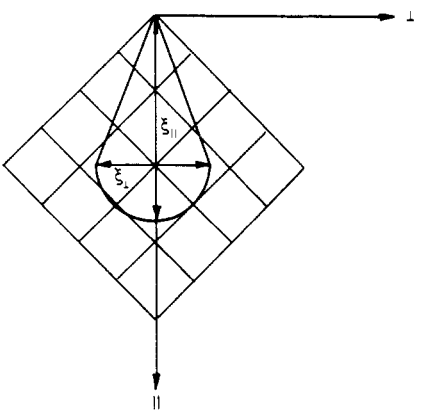

Figure 4. Cluster shape below the percolation threshold. $\xi_\parallel$ and $\xi_\perp$ are the parallel and perpendicular correlation lengths, respectively.

Let us restrict ourselves for the time being to the two-dimensional case. The best results for the isotropic case have been obtained by transfer matrix techniques (Kinzel and Yeomans 1981). Another situation of interest is the anisotropic case for which $p_x = 1$ and $p_y = p$, because this case is exactly solvable (Domany and Kinzel 1981). We shall consider these two cases below.

Let us now recall briefly how the RSRG transformation is defined for this problem (Kamphorst Leal da Silva and Droz 1985). We consider a symmetric $b \times b$ cell. Let

p_x and p_y, the probabilities for the presence of a directed bond in the x and y directions respectively (see figure 2). The first step of the transformation is to move the bonds along the x direction as shown in figure 2(b). The new probability $\tilde{p}_x$ is obtained by taking into account the contribution of all the oriented paths starting from the origin and ending on the line AB. The second step is to decimate along the y direction, leading to a new probability $\tilde{p}_y$. The third step is to repeat the first step for the $\tilde{p}_y$. Finally, the last step is to decimate along the x direction (see figure 2(c)). After the analytic continuation $b \to 1 + \delta b$, one finds the following recursion relations:

$$\begin{aligned} p_x' &= p_x + \delta b\left(p_x \ln p_x - \frac{p_x p_y q_x \ln[q_x p_y]}{1 - q_x p_y}\right) \\ p_y' &= p_y + \delta b\left(p_y \ln p_y - \frac{p_x p_y q_y \ln[p_x q_y]}{1 - p_x q_y}\right) \end{aligned} \tag{3.1}$$

where $q_j = 1 - p_j, j = x, y$. The non-trivial fixed points are the isotropic one $p_x = p_y = p^* = 0.6456$ and $p_y^* = 0.5$, $p_x^* = 1.0$ (Kamphorst Leal da Silva and Droz 1985). Defining the correlation length exponent ν through the relation:

$$\left.\frac{\mathrm{d}p_x'}{\mathrm{d}p_x}\right|_{p_x^*; p_y^*} = b^{1/\nu} \tag{3.2}$$

one finds $\nu = 1.286$ for the isotropic fixed point and $\nu = 1.629$ for the anisotropic one. But here again, it is not clear how this exponent ν is related to the exponents of interest $\nu_\parallel$ and $\nu_\perp$.

We have to implement the transformation by introducing two effective lengths into the problem. The two correlation lengths can be defined in terms of the moment of the pair connectedness function $\Gamma(0, j)$, defined as the probability that a pair of sites at the origin and at an arbitrary point j of the lattice are connected by a path of bonds that can be traversed in the direction of the arrows. Let $r_{0j,\parallel}$ and $r_{0j,\perp}$ be respectively the projections of the vector $\overrightarrow{0j}$ in the parallel and perpendicular directions. Then, one can define for $\alpha = \parallel$ or $\perp$:

$$\xi_\alpha^2(p) = \sum_j \Gamma(0, j) r_{0j,\alpha}^2 \left(\sum_j \Gamma(0, j)\right)^{-1}. \tag{3.3}$$

But

$$\xi_\alpha^2(p) \sim (p - p_c)^{-2\nu_\alpha}. \tag{3.4}$$

Thus, by computing ξ_α^2 for the original lattice and the rescaled one (b being the rescaling factor), one finds

$$\frac{\xi_\alpha^2(p^*; b)}{\xi_\alpha^2(p^*; 1)} = \left(\frac{\mathrm{d}p'}{\mathrm{d}p}\right)_{p^*}^{2\nu_\alpha}. \tag{3.5}$$

Moreover, in complete analogy with the DSAW case, one has in the critical regime

$$\frac{\xi_\alpha^2(p)}{\xi_\alpha^2(p')} = \frac{L_\alpha^2(p^*; b)}{L_\alpha^2(p^*; 1)} \tag{3.6}$$

where the two effective lengths correspond to the basic units of length of the original and renormalised cells. The problem is to compute those lengths following the steps of the renormalisation transformation. Referring to figure 2, we have the following situation. The original lattice Λ_1 with probabilities p_x and p_y in the directions x and

y has equal basic lengths in both directions, chosen arbitrarily to be unity. After the two first renormalisation group steps, the probabilities are $\tilde{p}_x$ and $\tilde{p}_y$ and the basic lengths of the lattice Λ_2 are $L_\alpha(\tilde{x}_\alpha)$ and $L_\alpha(\tilde{y}_\alpha)$ in the directions $\tilde{x}$ and $\tilde{y}$ respectively. Thus, according to (3.3), we can write

$$\frac{\Sigma_{j\in\Lambda_1} l_{\alpha j}^2 \Gamma(0;j)}{\Sigma_{j\in\Lambda_1}\Gamma(0;j)} \equiv \frac{A_{1\alpha}}{B_{1\alpha}} = \frac{\tilde{p}_x L_\alpha^2(\tilde{p}_x) + \tilde{p}_y L_\alpha^2(\tilde{p}_y)}{1+\tilde{p}_x+\tilde{p}_y} \tag{3.7}$$

where $l_{\alpha j}$ is the projection of the vector $\overline{0j}$ in the direction α, and

$$\frac{\Sigma_{j\in\Lambda_2} \tilde{l}_{\alpha j}^2 \tilde{\Gamma}(0;j)}{\Sigma_{j\in\Lambda_2}\tilde{\Gamma}(0;j)} \equiv \frac{A_{2\alpha}}{B_{2\alpha}} = \frac{p_x' L_\alpha^2(p_x') + p_y' L_\alpha^2(p_y')}{1+p_x'+p_y'} = \rho_b^\alpha(p_x';p_y') \tag{3.8}$$

where $\tilde{l}_{\alpha j}$ is the projection of the vector $\overline{0j}$ in the direction α measured in units of $L_\alpha(\tilde{x})$ and $L_\alpha(\tilde{y})$. Finally

$$\frac{\rho_b^\alpha(p_x^*;p_y^*)}{\rho_1^\alpha p_x^*;p_y^*)} = \frac{\xi_\alpha^2(b)}{\xi_\alpha^2(1)} = \left(\frac{\mathrm{d}p'}{\mathrm{d}p}\right)_{p^*}^{2\nu_\alpha}. \tag{3.9}$$

We can compute the different quantities of interest by inspection. One finds

$$\begin{aligned} A_{1\|} &= \tfrac{1}{2}(p_x+p_y)\sum_{n=0}^{b-1}(n+1)^2 p_y^n \\ B_{1\|} &= 1+(p_x+p_y)\sum_{n=0}^{b-1} p_y^n. \end{aligned} \tag{3.10}$$

Using (2.10), (2.11) and (2.26) and taking the analytic continuation $b \to 1+\delta b$, one finds

$$\begin{aligned} A_{1\|} &= \tfrac{1}{2}(p_x+p_y)\left[1+\delta b\left(2-\frac{2}{q_y^2}+\ln p_y - \frac{\ln p_y}{q_y^2} - \frac{2p_y \ln p_y}{q_y^3}\right)\right] \\ B_{1\|} &= (1+p_x+p_y)\left[1-\delta b\left(\frac{p_x+p_y}{1+p_x+p_y}\right)\frac{p_y \ln p_y}{q_y}\right]. \end{aligned} \tag{3.11}$$

It follows that

$$\begin{aligned} \tilde{p}_x L_\|^2(\tilde{p}_x) + \tilde{p}_y L_\|^2(\tilde{p}_y) = \tfrac{1}{2}(p_x+p_y)\Bigg[1+\delta b\Bigg(2-\frac{2}{q_y^2}+\ln p_y - \frac{\ln p_y}{q_y^2} - \frac{2p_y \ln p_y}{q_y^3} \\ + \frac{(p_x+p_y)p_y \ln p_y}{(1+p_x+p_y)q_y} + \frac{p_y \ln p_y}{(1+p_x+p_y)} - \frac{q_x p_x p_y \ln(p_y q_x)}{(1+p_x+p_y)(1-p_y q_x)}\Bigg)\Bigg]. \end{aligned} \tag{3.12}$$

The next transformation leads to

$$B_{2\|} = 1+(\tilde{p}_x+\tilde{p}_y)\sum_{n=0}^{b-1}\tilde{p}_x^n \tag{3.13}$$

and

$$\begin{aligned} A_{2\|} &= [\tilde{p}_y L_\|^2(\tilde{p}_y) + \tilde{p}_x\tilde{p}_y(L_\|(\tilde{p}_y)+L_\|(\tilde{p}_x))^2 \\ &\quad + \tilde{p}_y\tilde{p}_x^2(L_\|(\tilde{p}_y)+2L_\|(\tilde{p}_x))^2 + \ldots + \tilde{p}_y\tilde{p}_x^{b-1}[L_\|(\tilde{p}_y)+(b-1)L_\|(\tilde{p}_x)]^2] \\ &\quad + [\tilde{p}_x L_\|^2(\tilde{p}_x) + \tilde{p}_x^2(2L_\|(\tilde{p}_x))^2 + \ldots + \tilde{p}_x^b(bL_\|(\tilde{p}_x))^2] \\ &= \tilde{p}_y\sum_{n=0}^{b-1}(L_\|(\tilde{p}_y)+nL_\|(\tilde{p}_x))^2\tilde{p}_x^n + L_\|^2(\tilde{p}_x)\sum_{n=0}^{b} n^2\tilde{p}_x^n. \end{aligned} \tag{3.14}$$

In the limit $b \to 1+\delta b$, one finds using (2.10), (2.11), (2.26) and (3.13),

$$B_{2\|} = z\left[1+\delta b\left(\frac{p_y \ln p_y}{z} - \frac{p_x p_y q_x \ln(p_y q_x)}{z(1-p_y q_x)} - \frac{(p_x+p_y)p_x \ln p_x}{z q_x}\right)\right]$$

$$A_{2\|} = \tfrac{1}{2}(p_x+p_y)\left[1+\delta b\left(2-\frac{2}{q_y^2}+\ln p_y - \frac{\ln p_y}{q_y^2} - \frac{2p_y \ln p_y}{q_y^3} + \frac{(p_x+p_y)p_y \ln p_y}{z q_y} + \frac{p_y \ln p_y}{z}\right.\right.$$
$$\left.\left. - \frac{p_x p_y q_x \ln(p_y q_x)}{z(1-p_y q_x)}\right)\right] + \frac{\delta b}{2}\left[2p_x + p_x \ln p_x - \frac{p_x p_y \ln p_x}{q_x} - \frac{2p_x p_y}{q_x} - \frac{2p_x p_y \ln p_x}{q_x}\right.$$
$$\left. - \frac{2p_y p_x^2 \ln p_x}{q_x^2} - (1+p_y)\left(\frac{2p_x}{q_x^2} + \frac{p_x \ln p_x}{q_x^2} + \frac{2p_x^2 \ln p_x}{q_x^3}\right)\right] \tag{3.15}$$

where $z = 1+p_x+p_y$. Thus

$$\rho^{\|}_{1+\delta b}(p_x'; p_y') = \frac{1}{2}\frac{x+y}{z}\left[1+\delta b\left(2-\frac{2}{q_y^2}+\ln p_y - \frac{\ln p_y}{q_y^2} - \frac{2p_y \ln p_y}{q_y^3} + \frac{(p_x+p_y)p_y \ln p_y}{z q_y}\right)\right.$$
$$\left. + \delta b\left(2-\frac{2}{q_x^2}+\ln p_x - \frac{\ln p_x}{q_x^2} - \frac{2p_x \ln p_x}{q_x^3} + \frac{(p_x+p_y)p_x \ln p_x}{z q_x}\right)\right]. \tag{3.16}$$

Finally, using (3.8), one obtains

$$\frac{\xi_{\|}^2(1+\delta b)}{\xi_{\|}^2(1)} = 1+\delta b\left(2-\frac{2}{q_y^2}+\ln p_y - \frac{\ln p_y}{q_y^2} - \frac{2p_y \ln p_y}{q_y^3} + \frac{(p_x+p_y)}{z}\frac{p_y \ln p_y}{q_y}\right)$$
$$+ \delta b\left(2-\frac{2}{q_x^2}+\ln p_x - \frac{\ln p_x}{q_x^2} - \frac{2p_x \ln p_x}{q_x^3} + \frac{(p_x+p_y)}{z}\frac{p_x \ln p_x}{q_x}\right). \tag{3.17}$$

We can now compute the quantities for the transverse direction following the same lines. One finds

$$A_{1\perp} = \tfrac{1}{2}(p_x+p_y)\sum_{n=0}^{b-1}(n^2+1)p_y^n + (p_y-p_x)\sum_{n=0}^{b-1} n p_y^n$$

$$B_{1\perp} = B_{1\|} \qquad B_{2\perp} = B_{2\|} \tag{3.18}$$

$$A_{2\perp} = \tilde{p}_y L_\perp^2(\tilde{p}_y) + \tilde{p}_x L_\perp^2(\tilde{x}) - \delta b\left[\tfrac{1}{2}(p_x+p_y)\frac{p_x \ln p_x}{q_x} + \tfrac{1}{2}(p_x-p_y)\left(\frac{2p_x}{q_x} + \frac{2p_x \ln p_x}{q_x}\right.\right.$$
$$\left.\left. + \frac{2p_x^2 \ln p_x}{q_x^2}\right) + \tfrac{1}{2}(p_x+p_y)\left(\frac{2p_x}{q_x^2} + \frac{p_x \ln p_x}{q_x^2} + \frac{2p_x^2 \ln p_x}{q_x^3}\right)\right].$$

Thus

$$\frac{\xi_\perp^2(1+\delta b)}{\xi_\perp^2(1)} = 1 - \delta b\left[\frac{2p_y}{q_y^2} + \frac{p_y \ln p_y}{q_y} + \frac{p_y \ln p_y}{q_y^2} + \frac{2p_y^2 \ln p_y}{q_y^3} - \frac{(p_x+p_y)}{z}\frac{p_y \ln p_y}{q_y}\right.$$
$$+ \frac{(p_y-p_x)}{(p_x+p_y)}\left(\frac{2p_y}{q_y} + \frac{2p_y \ln p_y}{q_y} + \frac{2p_y^2 \ln p_y}{q_y^2}\right)$$
$$+ \frac{2p_x}{q_x^2} + \frac{p_x \ln p_x}{q_x} + \frac{p_x \ln p_x}{q_x^2} + \frac{2p_x^2 \ln p_x}{q_x^3}$$
$$\left. - \frac{(p_x+p_y)}{z}\frac{p_x \ln p_x}{q_x} + \frac{(p_x-p_y)}{(p_x+p_y)}\left(\frac{2p_x}{q_x} + \frac{2p_x \ln p_x}{q_x} + \frac{2p_x^2 \ln p_x}{q_x^2}\right)\right]. \tag{3.19}$$

Using (3.1) and (3.5), we can then extract the numerical values of the exponents. The results are the following. At the isotropic critical fixed point $\nu_{\parallel} = 1.76$ and $\nu_{\perp} = 0.66$ to be compared with the best estimate $\nu_{\parallel} = 1.73$ and $\nu_{\perp} = 1.1$. For the anisotropic critical fixed point one finds $\nu_{\parallel} = 2.26$ and $\nu_{\perp} = 0.88$ to be compared with the exact values $\nu_{\parallel} = 2.0$ and $\nu_{\perp} = 1.0$. The values obtained for $\nu_{\parallel}$ are rather good. The values for $\nu_{\perp}$ are less precise but qualitatively reasonable.

One of the merits of this method is the fact that it can be applied to the general d-dimensional case. The main steps of this calculation and the results are given in appendix 2.

4. Conclusions

The method used in the preceding sections mixed two ingredients. Firstly, the critical fixed points are obtained using an isotropic renormalisation group transformation. At each iteration of the transformation, the number of degrees of freedom of the system is reduced by a factor b^d. Secondly, two lengths defining two different rescaling $b_{\parallel}$ and $b_{\perp}$ for the basic lengths of the lattices are introduced. Are those two ingredients compatible? The minimum criterion of compatability to be fulfilled is that the reduction of the number of degrees of freedom should be the same in the two procedures. Thus

$$b^d = b_{\parallel} b_{\perp}^{(d-1)}. \tag{4.1}$$

This condition can be considered as a self-consistent condition allowing us to test the quality of the approximation used.

Let us look at the situation for the two-dimensional DSAW. In the first procedure, in which one computes the two effective lengths directly, one has (see equations (2.13) and (2.16))

$$\begin{aligned} b_{\parallel} &= 1 + \delta b 2 \ln 2 \\ b_{\perp} &= 1 + \delta b (2 - 2 \ln 2) \end{aligned} \tag{4.2}$$

and thus $b_{\parallel} b_{\perp} = b^2$.

For the second procedure, in which the effective lengths are defined as the square root of the average of the square effective lengths associated to the different paths, one finds (see (2.28) and (2.32)):

$$\begin{aligned} b_{\parallel} &= 1 + \delta b (5 \ln 2 - 2) \\ b_{\perp} &= 1 + \delta b \ln 2. \end{aligned} \tag{4.3}$$

Thus $b_{\parallel} b_{\perp} = 1 + \delta b (2.159)$ and not $1 + \delta b (2.0)$ as it should be according to (4.1). However, 2.159 is not too far from 2.0 and the self-consistent criterion is almost fulfilled.

However, by enforcing the self-consistent criterion (4.1), one has a way to compute $b_{\perp}$ knowing $b_{\parallel}$ or vice versa. Let us apply this strategy to the above situation. Assuming $b_{\perp}$ given by (4.3), one finds $b_{\parallel} = 1 + \delta b (2 - \ln 2)$. The corresponding exponents are

$$\nu_{\parallel} = 0.943 \qquad \nu_{\perp} = 0.5. \tag{4.4}$$

Thus for the DSAW problem, all the strategies considered give the exact critical fixed point, one correlation length exponent exactly and the correlation length exponent within a few per cent.

For the two-dimensional directed percolation, the situation is the following. At the isotropic fixed point, one has from (3.17) and (3.19):

$$\begin{aligned} b_{\parallel} &= 1+\delta b(1.369) \\ b_{\perp} &= 1+\delta b(0.513). \end{aligned} \tag{4.5}$$

Thus $b_{\parallel}b_{\perp}=1+1.88\delta b$ while $b^2=1+2\delta b$. At the anisotropic fixed point ($p_y^*=1$; $p_x^*=0.5$)

$$\begin{aligned} b_{\parallel} &= 1+\delta b(1.39) \\ b_{\perp} &= 1+\delta b(0.54). \end{aligned} \tag{4.6}$$

Thus $b_{\parallel}b_{\perp}=1+1.93\delta b$ while $b^2=1+2\delta b$. Thus in both cases the self-consistent condition (4.1) is almost satisfied. Keeping the result for $b_{\parallel}$ and determining $b_{\perp}$ through the condition (4.1) leads to $b_{\perp}=1+\delta b(0.631)$ for the isotropic fixed point, or

$$\nu_{\perp}=0.811 \tag{4.7}$$

while for the anisotropic fixed point, one finds $b_{\perp}=1+\delta b(0.631)$ and

$$\nu_{\perp}=0.999 \tag{4.8}$$

which is almost the exact value.

The transverse exponents $\nu_{\perp}$ are thus significantly improved by enforcing the self-consistent condition (4.1).

In conclusion, we see that our RSRG transformation implemented by the TEL hypothesis leads to a qualitatively satisfactory description of the directed geometrical transition. Both the location of the critical point as well as the values of the correlation length exponents can be found in arbitrary dimensions.

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Appendix 1

One considers here the case of d-dimensional directed self-avoiding walk. The longitudinal direction is given by one of the main diagonals of the hypercube defining the lattice. There are $(d-1)$ transverse directions in a plane perpendicular to the main diagonal.

The RSRG transformation for this general case is a direct extension of what has been done for two dimensions. Let p_j be the fugacity associated to a step in the direction j. The generalisation of the recursion relation (2.6) is then for $b\to 1+\delta b$:

$$p_j' = p_j + \delta b\left(p_j \ln p_j - \sum_{i\neq j=1}^{d} \frac{p_i p_j \ln p_i}{1-p_i}\right). \tag{A1.1}$$

For the isotropic case, $p_j = p\forall j$, the critical fixed point is thus

$$p^* = 1/d \tag{A1.2}$$

which is the exact value. Linearising the transformation around the isotropic fixed point leads to

$$\left.\frac{dp'}{dp'}\right|_{p^*} = 1 + \delta b \frac{d \ln d}{d-1}. \tag{A1.3}$$

The two effective lengths are introduced along the same lines as for two dimensions. Each path is projected on the longitudinal direction and on the perpendicular plane.

We restrict ourselves here to the symmetric case $p_j = p \forall j$. Using the generalisation of the first procedure described in § 2, one finds

$$\frac{L_\parallel(p; 1+\delta b)}{L_\parallel(p; 1)} = 1 + \delta b \left[1 + \ln p - (d-1)\left(\frac{p}{q} + \frac{2p \ln p}{q} + \frac{p^2 \ln p}{q^2} \right) \right] \tag{A1.4}$$

where $q = 1 - p$. Using (2.22) and the fixed point value (A1.2), one obtains

$$\nu_\parallel = 1 \qquad \forall d \tag{A1.5}$$

which is the exact result. For the perpendicular direction, one has

$$\frac{L_\perp(p; 1+\delta b)}{L_\perp(p; 1)} = 1 + \delta b \left(1 + \frac{p}{q} + \frac{1 - dpq}{q^2} \ln p \right). \tag{A1.6}$$

Using (2.21) and the fixed point value (A1.2), one finds

$$\nu_\perp = \frac{1}{\ln d} - \frac{1}{d-1} \tag{A1.7}$$

giving $\nu_\perp = 0.410$, 0.388 and 0.371 for $d = 3$, 4 and 5 respectively. The exact solution is $\nu_\perp = 0.5$ for all dimensions.

The correlation length exponents can be also computed by generalising the second procedure described in § 2. In this case, one finds at the isotropic fixed point

$$\frac{L_\parallel^2(p^*; 1+\delta b)}{L_\parallel^2(p^*; 1)} = 1 + \delta b \left(\frac{d(3d-1)}{(d-1)^2} \ln d - \frac{2d}{(d-1)} \right). \tag{A1.8}$$

Using (2.28) and (A1.3), one finds

$$\nu_\parallel = \frac{(3d-1)}{2(d-1)} - \frac{1}{\ln d}. \tag{A1.9}$$

Thus, $\nu_\parallel = 1.09$, 1.11 and 1.13 for $d = 3$, 4 and 5 respectively. As for the two-dimensional case, this prescription does not give the exact answer, but is a good approximation for d not too large.

For the perpendicular direction, one finds

$$\frac{L_\perp^2(p^*; 1+\delta b)}{L_\perp^2(p^*; 1)} = 1 + \delta b \left[2 + \frac{2}{d-1} \left(1 - \ln d - \frac{\ln d}{d-1} \right) - \frac{2}{(d-1)^3} \left(1 - d \ln d - \frac{2d \ln d}{d-1} \right) \right]. \tag{A1.10}$$

Using (2.29) and (A1.3), one finds

$$\nu_\perp = \frac{1}{\ln d} \left(1 - \frac{1}{(d-1)^2} \right) + \frac{1}{(d-1)^3} + \frac{1}{2(d-1)^2} - \frac{1}{d} - \frac{1}{d(d-1)}. \tag{A1.11}$$

Thus $\nu_\perp = 0.433$, 0.400 and 0.375 for $d = 3$, 4 and 5 respectively. The values of $\nu_\perp$ are somewhat better than the ones obtained with the first prescription. However, they do not agree with the exact result $\nu_\perp = 0.5$. One reason for that is the violation of the self-consistent condition (4.1) which becomes more severe as the dimensionality of the system increases.

Appendix 2

One considers here the case of the d-dimensional directed percolation. There are now one longitudinal and $(d-1)$ tranverse directions.

The d-dimensional RSRG transformation is performed along the same lines as for the two-dimensional case. One obtains in the limit $b \to 1 + \delta b$ (Kamphorst Leal da Silva and Droz 1985)

$$p_j' = p_j + \delta b p_j \left[(1-p_j) \sum_{i \neq j=1}^{d} \left(\frac{p_i \ln[p_i(1-p_j)]}{p_i(1-p_j)-1} \right) + \ln p_j \right]. \tag{A2.1}$$

The number of fixed points proliferates rapidly with the dimension and, for simplicity, we shall restrict ourselves here to the symmetric fixed points ($p_i = p \forall i$). Those fixed points are the solution of

$$p^* \left(\ln p^* + \frac{(d-1)p^*(1-p^*)\ln[p^*(1-p^*)]}{p^*(1-p^*)-1} \right) = 0. \tag{A2.2}$$

The relevant eigenvalue of the linearised transformation at the critical fixed point is

$$\lambda = 1 + \delta b \left[1 + \ln p^* + \frac{(d-1)p^*}{p^*q^*-1} \left(1 - 2p^* + (2q^* - p^*) \ln(p^*q^*) \right. \right.$$
$$\left. \left. - \frac{(1-2p^*)p^*q^* \ln(p^*q^*)}{p^*q^*-1} \right) \right]. \tag{A2.3}$$

Similarly to (3.4), the correlation length exponents $\nu_\parallel$ and $\nu_\perp$ are obtained as

$$\left. \frac{\xi_\alpha^2[\{p_j\}; 1+\delta b]}{\xi_\alpha^2[\{p_j\}; 1]} \right|_{p^*} = \left(\frac{dp'}{dp} \right)_{p^*}^{2\nu_\alpha} = \lambda^{2\nu_\alpha}. \tag{A2.4}$$

The two correlation lengths are computed along the same lines as for the two-dimensional case. After some tedious algebra, one finds

$$\left. \frac{\xi_\parallel^2[\{p_j\}; 1+\delta b]}{\xi_\parallel^2[\{p_j\}; 1]} \right|_{p^*} = 1 + \delta b d g(p^*) \tag{A2.5}$$

where

$$g(p^*) = 2 - \frac{2}{q^{*2}} + \ln p^* - \frac{\ln p^*}{q^{*2}} - 2\frac{p^* \ln p^*}{q^{*3}} + \frac{dp^*}{1+dp^*} \frac{p^* \ln p^*}{q^*}. \tag{A2.6}$$

Using (A2.3)-(A2.5) one can compute the exponent $\nu_\parallel$. In the limit $d \to \infty$, one finds the following asymptotic behaviour:

$$\nu_\parallel = \tfrac{7}{4} - 2/\ln d + 11/4d + O(1/d^2). \tag{A2.7}$$

This result does not agree with the mean-field prediction $\nu_\parallel = 1$, which holds for $d \leq d_u = 5$.

For the perpendicular direction, one obtains at the isotropic critical fixed point

$$\frac{\xi_\perp^2(p^*; 1+\delta b)}{\xi_\perp^2(p^*; 1)} = 1 + \delta b \left[\frac{d}{1+dp^*} \frac{p^* \ln p^*}{q^*} + d\left(\frac{2p^*}{q^{*2}} + \frac{p^* \ln p^*}{q^{*2}} + \frac{2p^{*2} \ln p^*}{q^{*3}} \right) \right] \tag{A2.8}$$

from which $\nu_\perp$ can be computed, using (A2.3) and (A2.5). In the limit $d \to \infty$, one finds the following asymptotic behaviour:

$$\nu_\perp = \tfrac{3}{4} - 1/\ln d - 1/d \ln d + 9/4d + O(1/d^2) \tag{A2.9}$$

which again does not agree with the mean-field prediction $\nu_\perp = 0.5$.

However, for d not too large, the values obtained for the exponents are more reasonable. For example, in three dimensions, one finds $\nu_\parallel = 1.45$ and $\nu_\perp = 0.55$, values to be compared with the best estimates $\nu_\parallel = 1.27$ and $\nu_\perp = 0.735$. As far as the percolation threshold is concerned, one finds for $d \to \infty$:

$$p^* = d^{-1} + d^{-2} + 2d^{-3} + O(d^{-4}). \tag{A2.10}$$

The leading term agrees with the exact value.

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